International Release of Mathematics and Mathematical Sciences

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Certain Classes of Multivalent Harmonic Functions Associated with p - repeated Integral operators

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Abstract

Making use of p - repeated integral operator in this paper we introduce a new class of complex- valued multivalent harmonic function. An equivalent convolution class condition and a sufficient coefficient condition for this class is obtained. It is proved that this coefficient condition is necessary for its subclass. Further, results on bounds, inclusion relation, extreme points, a convolution property and a result based on the integral operator are obtained.

Keywords Multivalent harmonic starlike (convex) functions, Erdélyi-Kober integral operator; Hohlov operator, Carlson and Shaffer operator, convolution, Wright generalized hypergeometric (Wgh) function, Gauss hypergeometric function, incomplete beta function.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real-valued harmonic in D. In any simply connected domain $D \subset C$, f can be written in the form: $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $\left|h'(z)\right| > \left|g'(z)\right|$ in D (see [8]). Let H denote a class of harmonic functions $f = h + \overline{g}$, which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

Duren, Hengartner and Laugesen [9] has given the concept of multivalent harmonic functions by proving argument principle for harmonic complex valued functions. Using this concept, Ahuja and Jahagiri [4], [5] introduced the family $H(m)$, $m \in N = (1,2,3.....)$ of all m -valent, harmonic and orientation preserving functions in the open disk $\Delta = \{z : |z| < 1\}$. A function f in $H(m)$ can be expressed as:

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are m-valent analytic functions in the open unit disk Δ of the form:

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n, g(z) = \sum_{n=m}^{\infty} b_n z^n, |b_m| < 1, m \in N = \{1,2,3....\}. \tag{1.2}$$

Whereas $TH(m)$ denote the subclass of functions $f = h + \overline{g} \in H(m)$ such that

$$h(z)=z^m-\sum_{n=m+1}^{\infty}a_n z^n,\ g(z)=\sum_{n=m}^{\infty}b_n z^n, |b_m|<1 \tag{1.3}$$

Recently, several fractional calculas operators have found their applications in geometric function theory. Many research papers [1, 2, 3] on harmonic functions defined by certain operators such as Dziok and Srivastava operator [10], Hohlov operator [16], Carlson and shaffer operator [7] have been published. The Wright's generalized hypergeometric (Wgh) function [13, 17] for positive real numbers $a_i\ (i=1,2,...,q), b_i (i=1,2,...,s)$ and for positive integers $A_i\ (i=1,2,...,q), B_i\ (i=1,2,...,s)$ with $1+\sum_{i=1}^{s}B_i-\sum_{i=1}^{q}A_i\geq 0$ is defined by

$${}_q\Psi_s\left(\begin{matrix}(a_i,A_i)_{1,q};\\(b_i,B_i)_{1,s};z\end{matrix}\right)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(a_i+A_i n)z^n}{\prod_{i=1}^{s}\Gamma(b_i+B_i n)n!}, \tag{1.4}$$

which is analytic in Δ if $q=s+1$.

In particular, if $A_1=...=A_q=B_1=...=B_s=1$,

$${}_q\Psi_s\left(\begin{matrix}(a_i,1)_{1,q};\\(b_i,1)_{1,s};z\end{matrix}\right)=\frac{\prod_{i=1}^{q}\Gamma(a_i)}{\prod_{i=1}^{s}\Gamma(b_i)}\ {}_qF_s((a_i)_{1,q};(b_i)_{1,s};z), \tag{1.5}$$

where ${}_qF_s((a_i)_{1,q};(b_i)_{1,s};z)\equiv{}_qF_s(a_1,...a_q;b_1,...b_s;z)$ is the generalized hypergeometric (gh) function defined by

$${}_qF_s((a_i)_{1,q};(b_i)_{1,s};z)=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{q}(a_i)_n z^n}{\prod_{i=1}^{s}(b_i)_n n!}. \tag{1.6}$$

The symbol $(\lambda)_n$ is called Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\lambda(\lambda+1)...(\lambda+n-1).$$

The Hadmard product (convolution) '$*$' of two power series converging in Δ is defined by

$$\sum_{n=0}^{\infty}a_n z^n*\sum_{n=0}^{\infty}b_n z^n=\sum_{n=0}^{\infty}a_n b_n z^n.$$

The Erdélyi-Kober integral operator [13] $I_\beta^{\nu,\delta}$, is defined for $\beta\in\mathbb{R}_+$, $\nu\in\mathbb{R}$ by

$$I_\beta^{\nu,0}h(z)=h(z),$$

$$I_\beta^{\nu,\delta}h(z)=\frac{1}{\Gamma(\delta)}\int_0^1(1-t)^{\delta-1}t^\nu h(zt^{\frac{1}{\beta}})dt, \delta>0.$$

With the help of the integral operator $I_\beta^{\nu,\delta}$, an p-repeated integral operator $I_{\beta_i,p}^{(\nu_i),(\delta_i)}$ [14], [15] for analytic functions is defined as follows:

Let h be an analytic function defined in Δ, for $\beta_i \in \mathbb{R}_+$, $\delta_i \in \mathbb{R}_+ \cup \{0\}$, $\nu_i \in \mathbb{R}$, $i = 1,2,\ldots,p$, an p-repeated integral operator is defined by

$$I_{\beta_1,1}^{\nu_1,0} h(z) = h(z),$$

$$I_{\beta_1,1}^{\nu_1,\delta_1} h(z) \equiv I_{\beta_1}^{\nu_1,\delta_1} h(z)$$

$$= \frac{1}{\Gamma(\delta_1)} \int_0^1 (1-t)^{\delta_1 - 1} t^{\nu_1} h(zt^{\frac{1}{\beta_1}}) dt, \delta_1 > 0,$$

$$I_{\beta_i,2}^{(\nu_i),(0)} h(z) = h(z),$$

$$I_{\beta_i,2}^{(\nu_i),(\delta_i)} h(z) = \prod_{i=1}^{2} I_{\beta_i}^{\nu_i,\delta_i} h(z)$$

$$= I_{\beta_2}^{\nu_2,\delta_2} \left[I_{\beta_1}^{\nu_1,\delta_1} h(z) \right], \delta_1 + \delta_2 > 0,$$

and for $p \in \mathbb{N} = \{1,2,3,\ldots\}$,

$$I_{\beta_i,p}^{(\nu_i),(0)} h(z) = h(z), \tag{1.7}$$

$$I_{\beta_i,p}^{(\nu_i),(\delta_i)} h(z) = \prod_{i=1}^{p} I_{\beta_i}^{\nu_i,\delta_i} h(z), \sum_{i=1}^{p} \delta_i > 0.$$

The image of power function z^n [14, 15], under the operator $I_{\beta_i,p}^{(\nu_i),(\delta_i)}$, defined in (1.7), is given by $I_{\beta_i,p}^{(\nu_i),(\delta_i)} z^n = \lambda_n z^n$, (1.8)

Where
$$\lambda_n := \prod_{i=1}^{p} \frac{\Gamma\left(\nu_i + 1 + \frac{n}{\beta_i}\right)}{\Gamma\left(\nu_i + \delta_i + 1 + \frac{n}{\beta_i}\right)}, \tag{1.9}$$

for each $n > \max_{1 \le i \le p} [-\beta_i(\nu_i + 1)]$.

Involving p-repeated integral operators of the form (1.7), with the use of (1.8), an operator W on the class $H(m)$ is defined as follows:

2.Definition Let $f = h + \overline{g}$ be given by (1.1), for $p \in \mathbb{N} = \{1,2,3,\ldots\}$, $\beta_i, \beta_i^{'} \in \mathbb{R}_+$, $\delta_i, \delta_i^{'} \in \mathbb{R}_+ \cup \{0\}$, $\nu_i, \nu_i^{'} \ge -1$, $i = 1,2,\ldots,p$, an operator $W \equiv W\begin{bmatrix} (\nu_i),(\delta_i),(\nu_i^{'}),(\delta_i^{'}), \\ \beta_i, \beta_i^{'}, p \end{bmatrix} : H(m) \to H(m)$ is defined by

$$Wf(z) = \frac{1}{\lambda_m} I_{\beta_i,p}^{(\nu_i),(\delta_i)} h(z) + \frac{1}{\lambda_m^{'}} \overline{I_{\beta_i^{'},p}^{(\nu_i^{'}),(\delta_i^{'})} g(z)}, \tag{1.10}$$

where for any $n \ge m$, λ_n is given by (1.9) and

$$\lambda_n^{'} := \prod_{i=1}^{p} \frac{\Gamma\left(\nu_i^{'} + 1 + \frac{n}{\beta_i^{'}}\right)}{\Gamma\left(\nu_i^{'} + \delta_i + 1 + \frac{n}{\beta_i^{'}}\right)}. \tag{1.11}$$

The series representation of $Wf(z),$ defined in(1.10) is given by

$$Wf(z) = z^m + \sum_{n=m+1}^{\infty} \frac{\lambda_n}{\lambda_m} a_n z^n + \overline{\sum_{n=m}^{\infty} \frac{\lambda_n^{'}}{\lambda_m^{'}} b_n z^n}, \tag{1.12}$$

where $\lambda_n, \lambda_n^{'}, n \geq m$ are given by (1.9) and (1.11) respectively. We see that $Wf(z)$ given by (1.12) can also be expressed as a convolution of two functions belonging to $H(m)$ class by

$$Wf(z) = z^m + \sum_{m=n+1}^{\infty} \frac{\lambda_n}{\lambda_m} z^n * \sum_{m=n+1}^{\infty} a_n z^n + \overline{\sum_{n=m}^{\infty} \frac{\lambda_n^{'}}{\lambda_m^{'}} z^n * \sum_{n=m}^{\infty} b_n z^n}$$

$$= \left(\frac{z^m}{\lambda_m} \Psi_1(z)\right) * h(z) + \overline{\left(\frac{z^m}{\lambda_m^{'}} \Psi_1^{'}(z)\right) * g(z)},$$

where

$$\Psi_1(z) \equiv {}_{p+1}\Psi_p \left(\begin{matrix} (1,1), \left(\nu_i + 1 + \frac{m}{\beta_i}, \frac{1}{\beta_i}\right)_{1,p}; \\ \left(\nu_i + \delta_i + 1 + \frac{m}{\beta_i}, \frac{1}{\beta_i}\right)_{1,p}; z \end{matrix} \right), \text{ and } \Psi_1^{'}(z) \equiv {}_{p+1}\Psi_p \left(\begin{matrix} (1,1), \left(\nu_i^{'} + 1 + \frac{m}{\beta_i^{'}}, \frac{1}{\beta_i^{'}}\right)_{1,p}; \\ \left(\nu_i^{'} + \delta_i^{'} + 1 + \frac{m}{\beta_i^{'}}, \frac{1}{\beta_i^{'}}\right)_{1,p}; z \end{matrix} \right)$$

are Wgh functions and $\lambda_n, \lambda_n^{'}, n \geq m$ are given by (1.9) and (1.11) respectively. In general, we denote Wgh functions

$$\Psi_k(z) := {}_{p+1}\Psi_p \left(\begin{matrix} (k,1), \left(\nu_i + 1 + \frac{(m+k-1)}{\beta_i}, \frac{1}{\beta_i}\right)_{1,p}; \\ \left(\nu_i + \delta_i + 1 + \frac{(m+k-1)}{\beta_i}, \frac{1}{\beta_i}\right)_{1,p}; z \end{matrix} \right),$$

$$\Psi_k^{'}(z) := {}_{p+1}\Psi_p \left(\begin{matrix} (k,1), \left(\nu_i^{'} + 1 + \frac{(m+k-1)}{\beta_i^{'}}, \frac{1}{\beta_i^{'}}\right)_{1,p}; \\ \left(\nu_i^{'} + \delta_i^{'} + 1 + \frac{(m+k-1)}{\beta_i^{'}}, \frac{1}{\beta_i^{'}}\right)_{1,p}; z \end{matrix} \right).$$

for $k = 1,2,3\ldots$. This Wgh functions Involving p -repeated integral operators for harmonic multivalent functions was widely discussed in [20].

Remark 1 Taking $\beta_i = 1 = \beta_i^{'},$ $\nu_i = a_i - 1 - m,$ $\nu_i^{'} = c_i - 1 - m,$ $\delta_i = b_i - a_i,$ $\delta_i^{'} = d_i - c_i$ for $i = 1,2,\ldots,p,$ the operator $Wf(z)$ defined by (1.10) reduces to the operator $\Omega f(z)$ which is Dziok -Srivastava type operator involving generalized hypergeometric functions ${}_{p+1}F_p$ and is defined on $H(m)$ by

$$\Omega f(z) := \prod_{i=1}^{p}\frac{\Gamma(b_i)}{\Gamma(a_i)} I_{1,p}^{(\nu_i),(\delta_i)} h(z) + \overline{\prod_{i=1}^{p}\frac{\Gamma(d_i)}{\Gamma(c_i)} I_{1_i,p}^{(\nu_i'),(\delta_i')} g(z)} \tag{1.13}$$

$$= z^m F_1(z) * h(z) + \overline{z^m F_1'(z) * g(z)},$$

Where $F_1(z) \equiv {}_{p+1}F_p(1,(a_i)_{1,p};(b_i)_{1,p};z)$, $F_1'(z) \equiv {}_{p+1}F_p'((1,(c_i)_{1,p};(d_i)_{1,p};z)$.

Remark 2 If we take, $p = 2$,
$\nu_1 = a_1 - 1 - m, \nu_2 = b_1 - 1 - m, \delta_1 = 1 - a_1, \delta_2 = c_1 - b_1; \nu_1' = a_2 - 1 - m, \nu_2' = b_2 - 1 - m$,
$\delta_1' = 1 - a_2, \delta_2' = c_2 - b_2$ and $\beta_i = 1 = \beta_i'$ $(i = 1,2)$, the operator $Wf(z)$ defined by (1.10) reduces to the operator $\mathrm{H}f(z)$ which is Hohlov type operator involving Gauss hypergeometric functions ${}_2F_1$ and is defined on $H(m)$ by

$$\mathrm{H}f(z) := \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(b_1)} I_{1,2}^{(\nu_i),(\delta_i)} h(z) + \overline{\frac{\Gamma(c_2)}{\Gamma(a_2)\Gamma(b_2)} I_{1,2}^{(\nu_i'),(\delta_i')} g(z)} \tag{1.14}$$

$$= z^m\, {}_2F_1(a_1,b_1;c_1;z) * h(z) + \overline{z^m\, {}_2F_1'(a_2,b_2;c_2;z) * g(z)}.$$

Remark 3 Taking $p = 1, \nu = a_1 - 1 - m, \delta = c_1 - a_1, \nu' = a_2 - 1 - m$ $,\delta' = c_2 - b_2$ and $\beta_i = 1 = \beta_i'$ the operator $Wf(z)$ defined by (1.10) reduces to $\mathrm{L}f(z)$ which is Carlson Shaffer type operator involving incomplete beta functions and is defined on $H(m)$ by

$$\mathrm{L}f(z) = \frac{\Gamma(c_1)}{\Gamma(a_1)} I_{1,1}^{(a_1-1-p),(c_1-a_1)} h(z) + \overline{\frac{\Gamma(c_2)}{\Gamma(a_2)} I_{1,1}^{(a_2-1-m),(c_2-b_2)} g(z)} \tag{1.15}$$

$$= z^m\, {}_2F_1(1,a_1;c_1;z) * h(z) + \overline{z^m\, {}_2F_1'(1,a_2;c_2;z) * g(z)}$$

For the purpose of this paper, we define a class $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ of functions $f \in H(m)$ if it satisfy the condition

$$\Re\left\{(1-\mu)\frac{Wf(z)}{z^m} + \mu(1-t)\frac{(Wf(z))'}{(z^m)'} + \mu t\frac{(Wf(z))''}{(z^m)''}\right\} > \frac{\gamma}{m} \tag{1.16}$$

where $\mu \ge 0$, $0 \le t \le 1$, $0 \le \gamma < m$, and $z = re^{i\theta}(r < 1, \theta \in \mathrm{R})$, $z' = \frac{\partial z}{\partial \theta}, z'' = \frac{\partial^2 z}{\partial \theta^2}, (Wf(z))' = \frac{\partial}{\partial \theta}(Wf(z))$ and $(Wf(z))'' = \frac{\partial^2}{\partial \theta^2}(Wf(z))$

It is special intrest beacuse for suitable choices of different operators defind in Remark (1-3) by taking some particular values of parameters, $p, \nu_i, \nu_i', \delta_i, \delta_i', \beta_i, \beta_i'$ we can define the following subclasses.

1.Taking $\Omega f(z)$ given by (1.13) in place of $Wf(z)$ defined by (1.10), we can defined a class $\Omega_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ which is emerge from class $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ (1.16) satisfying the criteria

$$\Re\left\{(1-\mu)\frac{\Omega f(z)}{z^m}+\mu(1-t)\frac{(\Omega f(z))'}{(z^m)'}+\mu t\frac{(\Omega f(z))''}{(z^m)''}\right\}>\frac{\gamma}{m}$$

where $\Omega f(z)$ is Dziok -Srivastava operator [11]. $\mu \ge 0$, $0 \le t \le 1$, $0 \le \gamma < m$ and $z = re^{i\theta}(r<1, \theta \in R)$, $z' = \frac{\partial z}{\partial \theta}, z'' = \frac{\partial^2 z}{\partial \theta^2}, (\Omega f(z))' = \frac{\partial}{\partial \theta}(\Omega f(z))$ and $(\Omega f(z))'' = \frac{\partial^2}{\partial \theta^2}(\Omega f(z))$.

2. Taking $Hf(z)$ given by (1.14) in place of $Wf(z)$ defined by (1.10) ,we can defined a class $H_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ which is emerge from class $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ (1.16) satisfying the criteria

$$\Re\left\{(1-\mu)\frac{Hf(z)}{z^m}+\mu(1-t)\frac{(Hf(z))'}{(z^m)'}+\mu t\frac{(Hf(z))''}{(z^m)''}\right\}>\frac{\gamma}{m}$$

where $Hf(z)$ is Hohlov operator [16]. $\mu \ge 0$, $0 \le t \le 1$, $0 \le \gamma < m$ and $z = re^{i\theta}(r<1, \theta \in R)$, $z' = \frac{\partial z}{\partial \theta}, z'' = \frac{\partial^2 z}{\partial \theta^2}, (Hf(z))' = \frac{\partial}{\partial \theta}(Hf(z))$ and $(Hf(z))'' = \frac{\partial^2}{\partial \theta^2}(Hf(z))$.

3. Taking $Lf(z)$ given by (1.14) in place of $Wf(z)$ defined by (1.10), we can defined a class $L_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ which is emerge from class $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ (1.16) satisfying the criteria

$$\Re\left\{(1-\mu)\frac{Lf(z)}{z^m}+\mu(1-t)\frac{(Lf(z))'}{(z^m)'}+\mu t\frac{(Lf(z))''}{(z^m)''}\right\}>\frac{\gamma}{m}$$

where $Lf(z)$ is Carlson Shaffer type operator [7]. $\mu \ge 0$, $0 \le t \le 1$, $0 \le \gamma < m$ and $z = re^{i\theta}(r<1, \theta \in R)$, $z' = \frac{\partial z}{\partial \theta}, z'' = \frac{\partial^2 z}{\partial \theta^2}, (Lf(z))' = \frac{\partial}{\partial \theta}(Lf(z))$ and $(Lf(z))'' = \frac{\partial^2}{\partial \theta^2}(Lf(z))$.

Based on some particular values of μ and t, where $\mu \ge 0, 0 \le t \le 1$, $0 \le \gamma < m$, the family $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ produces a passage from the class of harmonic functions:

1. $A_m^p([(\nu_i),(\delta_i),\beta_i];\gamma,t) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;0,t)$, consisting of functions f where

$$\Re\left\{\frac{Wf(z)}{z^m}\right\}>\frac{\gamma}{m},\ 0 \le \gamma < m. \tag{1.17}$$

2. $B_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;1,t)$, consisting of functions f where

$$\Re\left\{(1-t)\frac{(Wf(z))'}{(z^m)'}+t\frac{(Wf(z))''}{(z^m)''}\right\}>\frac{\gamma}{m},\ 0 \le t \le 1, 0 \le \gamma < m. \tag{1.18}$$

3. $C_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,0)$, consisting of functions f where

$$\Re\left\{(1-\mu)\frac{Wf(z)}{z^m}+\mu\frac{(Wf(z))'}{(z^m)'}\right\} > \frac{\gamma}{m},\ \mu \geq 0, 0 \leq \gamma < m. \tag{1.19}$$

4. $D_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,1)$, consisting of functions f where

$$\Re\left\{(1-\mu)\frac{Wf(z)}{z^m}+\mu\frac{(Wf(z))''}{(z^m)''}\right\} > \frac{\gamma}{m},\ \mu \geq 0, 0 \leq \gamma < m. \tag{1.190}$$

5. $E_m^p([(\nu_i),(\delta_i),\beta_i];\gamma) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;0,1)$, consisting of functions f where

$$\Re\left\{\frac{(Wf(z))'}{(z^m)'}\right\} > \frac{\gamma}{m},\ 0 \leq \gamma < m. \tag{1.191}$$

6. $F_m^p([(\nu_i),(\delta_i),\beta_i];\gamma) = R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;1,1)$, consisting of functions f where

$$\Re\left\{\frac{(Wf(z))''}{(z^m)''}\right\} > \frac{\gamma}{m}, 0 \leq \gamma < m. \ . \tag{1.192}$$

Several sub-classes defined above by taking particular values of μ and t on harmonic functions involving certain linear operator have recently been studied in [6, 18, 12, 19, 21] etc.
In this paper, an equivalent convolution class condition is derived and a coefficient inequality is obtained for the functions $f = h + \overline{g} \in H(m)$ to be in the class $R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$. It is also proved that this inequality is necessary for $f = h + \overline{g}$ to be in $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ class. Further, based on the coefficient inequality, results on bounds,inclusion relations, extreme points, convolution and convex combination and on an integral operator are obtained.

2 .Coefficient Inequality

Theorem 1 Let $\mu \geq 0$, $0 \leq t \leq 1$, $0 \leq \gamma < m, m \in \mathbb{N}$. If the function $f = h + \overline{g} \in H(m)$ (where h and g are of the form (1.2)) satisfies

$$\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n| + \sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n| \leq 1, \tag{2.1}$$

then f is sense-preserving, harmonic multivalent in Δ and $f \in R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$.

Proof. Under the given parametric constraints, we have

$$\frac{n}{m} \leq \frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m} \text{ and } \frac{n}{m} \leq \frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}, n \geq m. \tag{2.2}$$

Thus, for $f = h + \overline{g} \in H(m)$, where h and g are of the form (1.2), we get

$$\left|h'(z)\right| \ge m|z|^{m-1} - \sum_{n=m+1}^{\infty} n|a_n||z|^{n-1} \ge m|z|^{m-1}\left[1-\sum_{n=m+1}^{\infty}\frac{n}{m}|a_n|\right]$$

$$\ge m|z|^{m-1}\left[1-\sum_{n=m+1}^{\infty}\frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|\right]$$

$$\ge m|z|^{m-1}\left[\sum_{n=m}^{\infty}\frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)}\frac{\lambda'_n}{\lambda'_m}|b_n|\right] > \sum_{n=m}^{\infty} n|b_n||z|^{n-1} \ge \left|g'(z)\right|$$

which proves that $f(z)$ is sense preserving in Δ. Now to show that $f \in R_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, we need to show (1.16), that is

$$\Re\left\{(1-\mu)\frac{Wf(z)}{z^m}+\mu(1-t)\frac{(Wf(z))'}{(z^m)'}+\mu t\frac{(Wf(z))''}{(z^m)''}\right\} > \frac{\gamma}{m}, z \in \Delta, \tag{2.3}$$

Suppose $A(z) = \Re\left\{(1-\mu)\frac{Wf(z)}{z^m}+\mu(1-t)\frac{(Wf(z))'}{(z^m)'}+\mu t\frac{(Wf(z))''}{(z^m)''}\right\} > \frac{\gamma}{m}$

It is suffices to show that $\left|\frac{A(z)-1}{A(z)-\frac{2\gamma}{m}+1}\right| < 1$

Series expansion of $A(z)$ is given by

$$A(z) = 1+\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left\{1+\mu\left(\frac{n}{m}-1\right)\left(\frac{tn}{m}+1\right)\right\}a_n z^{n-m}+\sum_{n=m}^{\infty}\frac{\lambda'_n}{\lambda'_m}\left\{1+\mu\left(\frac{n}{m}+1\right)\left(\frac{tn}{m}-1\right)\right\}b_n \bar{z}^{-n} z^{-m}$$

and we have $\left|A(z)-\frac{2\gamma}{m}+1\right| - |A(z)-1|$

$$= \left|2(1-\frac{\gamma}{m})+\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left\{1+\mu\left(\frac{n}{m}-1\right)\left(\frac{tn}{m}+1\right)\right\}a_n z^{n-m}+\sum_{n=m}^{\infty}\frac{\lambda'_n}{\lambda'_m}\left\{1+\mu(\frac{n}{m}+1)(\frac{tn}{m}-1)\right\}b_n \bar{z}^{-n} z^{-m}\right|$$

$$-\left|\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left\{1+\mu\left(\frac{n}{m}-1\right)\left(\frac{tn}{m}+1\right)\right\}a_n z^{n-m}+\sum_{n=m}^{\infty}\frac{\lambda'_n}{\lambda'_m}\left\{1+\mu(\frac{n}{m}+1)(\frac{tn}{m}-1)\right\}b_n \bar{z}^{-n} z^{-m}\right|$$

$$\ge \frac{1}{m}\left[2(m-\gamma)-\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left|m+\mu(n-m)\left(\frac{tn}{m}+1\right)\right||a_n||z^{n-m}|\right.$$

$$-\sum_{n=m}^{\infty}\frac{\lambda'_n}{\lambda'_m}\left|m+\mu(n+m)\left(\frac{tn}{m}-1\right)\right||b_n||\bar{z}^{-n}||z^{-m}|$$

$$\left.-\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left|m+\mu(n-m)\left(\frac{tn}{m}+1\right)\right||a_n||z^{n-m}| - \sum_{n=m}^{\infty}\frac{\lambda'_n}{\lambda'_m}\left|m+\mu(n+m)\left(\frac{tn}{m}-1\right)\right||b_n||\bar{z}^{-n}||z^{-m}|\right]$$

$$= \frac{1}{m}\left[2(m-\gamma)-2\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left|m+\mu(n-m)\left(\frac{tn}{m}+1\right)\right||a_n||z^{n-m}|\right.$$

$$-2\sum_{n=m}^{\infty}\frac{\lambda_n^{'}}{\lambda_m^{'}}\left|m+\mu(n+m)\left(\frac{tn}{m}-1\right)\right||b_n||\overline{z}^{-n}||z^{-m}|\Bigg]$$

≥ 0
by (2.1) when $z = r \to 1$ and this proves Theorem 1.

We next show that the above sufficient coefficient condition is also necessary for functions in the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$

Theorem 2 Let $\mu \geq 0$, $0 \leq t \leq 1$, $0 \leq \gamma < m, m \in \mathbb{N}$. and let the function $f = h + \overline{g} \in \widetilde{H}(m)$ be such that h and g are given by (1.3). Then $f \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ if and only if (2.1) holds. The inequality (2.1) is sharp for the function given by

$$f(z) = z^m - \sum_{n=m+1}^{\infty}\frac{m(m-\gamma)}{|m^2+\mu(n-m)(tn+m)|\frac{\lambda_n}{\lambda_m}}|x_n|z^n \tag{2.4}$$

$$+\sum_{n=m}^{\infty}\frac{m(m-\gamma)}{|m^2+\mu(n+m)(tn-m)|\frac{\lambda_n^{'}}{\lambda_m^{'}}}|y_n|\overline{z^n},$$

$$\sum_{n=m+1}^{\infty}|x_n|+\sum_{n=m}^{\infty}|y_n| = 1.$$

Proof. The if part, follows from Theorem 1. To prove the "only if part" let $f = h + \overline{g} \in \widetilde{H}(m)$ be such that h and g are given by (1.3) and $f \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ then for $z = re^{i\theta}$ in Δ we obtain

$$\Re\left\{(1-\mu)\frac{Wf(z)}{z^m}+\mu(1-t)\frac{(Wf(z))^{'}}{(z^m)^{'}}+\mu t\frac{(Wf(z))^{''}}{(z^m)^{''}}\right\} > \frac{\gamma}{m}$$

$$=\Re\left\{(1-\mu)\frac{\frac{1}{\lambda_m}I_{\beta_i,p}^{(\nu_i),(\delta_i)}h(z)+\frac{1}{\lambda_m^{'}}\overline{I_{\beta_i^{'},p}^{(\nu_i^{'}),(\delta_i^{'})}g(z)}}{z^m}+\mu(1-t)\frac{z\left(\frac{1}{\lambda_m}I_{\beta_i,p}^{(\nu_i),(\delta_i)}h(z)\right)^{'}-z\left(\frac{1}{\lambda_m^{'}}\overline{I_{\beta_i^{'},p}^{(\nu_i^{'}),(\delta_i^{'})}g(z)}\right)^{'}}{mz^m}\right\}$$

$$+\Re\left\{\mu t\left(\frac{z^2\left(\frac{1}{\lambda_m}I_{\beta_i,p}^{(\nu_i),(\delta_i)}h(z)\right)^{''}+z\left(\frac{1}{\lambda_m}I_{\beta_i,p}^{(\nu_i),(\delta_i)}h(z)\right)}{m^2z^m}+\frac{z^2\left(\frac{1}{\lambda_m^{'}}\overline{I_{\beta_i^{'},p}^{(\nu_i^{'}),(\delta_i^{'})}g(z)}\right)^{''}+z\left(\frac{1}{\lambda_m^{'}}\overline{I_{\beta_i^{'},p}^{(\nu_i^{'}),(\delta_i^{'})}g(z)}\right)^{'}}{m^2z^m}\right)\right\}$$

$$\geq 1-\sum_{n=m+1}^{\infty}\frac{\lambda_n}{\lambda_m}\left|1+\mu\left(\frac{n}{m}-1\right)\left(\frac{tn}{m}+1\right)\right||a_n||z^{n-m}|-\sum_{n=m}^{\infty}\frac{\lambda_n'}{\lambda_m'}\left|1+\mu\left(\frac{n}{m}+1\right)\left(\frac{tn}{m}-1\right)\right||b_n||\bar{z}^{n}||z^{-m}|$$

$$>\frac{\gamma}{m}$$

The above inequality must hold for all $z\in\Delta$. in particular $z=r\to 1$ yields the required condition (2.1). Sharpness of the result can easily be verified for the function given by (2.4).

As a special case of Theorem 2, we obtain the following corollaries.

Corollary 1 For class (1.17) we can write, $f=h+\bar{g}\in\tilde{A}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma,t)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{m}{(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{m}{(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

Corollary 2 For class (1.18) we can write, $f=h+\bar{g}\in\tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma,t)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{|m^2+(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

Corollary 3 For class (1.19) we can write, $f=h+\bar{g}\in\tilde{C}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)m|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)m|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

Corollary 4 For class (1.190) we can write, $f=h+\bar{g}\in\tilde{D}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n^2-m^2)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+\mu(n^2-m^2)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

Corollary 5 For class (1.191) we can write, $f=h+\bar{g}\in\tilde{E}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{|n|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|-n|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

Corollary 6 For class (1.192) we can write, $f=h+\bar{g}\in\tilde{F}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma)$ if and only if

$$\sum_{n=m+1}^{\infty}\frac{n^2}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{n^2}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\leq 1,\text{ holds.}$$

3 Inclusion Relation

The inclusion relations between the classes $\tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t)$ and $\tilde{A}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma,t)$ for different values of μ are not so obvious. In this section we discuss the inclusion relation between above mentioned classes.

Theorem 3 for $n\in\{1,2,3..\}$ and $0\leq\gamma<m,$ we have

(i) $\tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t)\subset\tilde{A}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma,t)$

(ii) $\tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t)\subset \tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t), 0\le\mu\le 1$

(iii) $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)\subset \tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t), \mu\ge 1$

Proof. (i) Let $f(z)\in \tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t)$. in view of corollaries 1 and 2, we have

$$\sum_{n=m+1}^{\infty}\frac{m}{(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{m}{(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|$$

$$\le \sum_{n=m+1}^{\infty}\frac{|m^2+(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\le 1$$

(ii)Let $f(z)\in \tilde{B}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;t)$.For $0\le\mu\le 1$, we can write

$$\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|$$

$$\le \sum_{n=m+1}^{\infty}\frac{|m^2+(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\le 1$$

by corollary 2 and (ii) follows from Theorem 2

(iii) By the Theorem 2, if $\mu\ge 1$, we have

$$\sum_{n=m+1}^{\infty}\frac{|m^2+(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|$$

$$\le \sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n|+\sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\le 1$$

Therefore the result follows from corollary 2.

4 .Bounds

Our next theorems provide the bounds for the function in the class $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ which are followed by a covering result for this class.

Theorem 4 Let $\mu\ge 0$, $0\le t\le 1$, $0\le\gamma<m, m\in\mathbb{N}$. if $f=h+\bar{g}\in\tilde{H}(m)$, where h and g are of the form (1.3) belongs to the class $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then for $|z|=r<1$,

$$|Wf(z)|\le(1+|b_m|)r^m+\frac{mr^{m+1}}{m+1}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right), \tag{4.1}$$

And $$|Wf(z)|\ge(1-|b_m|)r^m-\frac{m}{m+1}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)r^{m+1}. \tag{4.2}$$

The result is sharp.

Proof. Let $f\in\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then on using (2.1), related to (1.3), by (1.10), we get for $|z|=r<1$,

$$|Wf(z)| \le (1+|b_m|)r^m + \sum_{n=m+1}^{\infty}\left(\frac{\lambda_n}{\lambda_m}|a_n| + \frac{\lambda_n^{'}}{\lambda_m^{'}}|b_n|\right)r^n$$

$$\le (1+|b_m|)r^m + r^{m+1}\sum_{n=m+1}^{\infty}\left(\frac{\lambda_n}{\lambda_m}|a_n| + \frac{\lambda_n^{'}}{\lambda_m^{'}}|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{m+1}\left(\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n| + \sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n^{'}}{\lambda_m^{'}}|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{m+1}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)$$

which proves the result (4.1). The result (4.2) can similarly be obtained. The bounds (4.1) and (4.2) are sharp for the function given by

$$f(z) = z^m + |b_m|\overline{z^m} + \frac{m}{(m+1)\frac{\lambda_{m+1}^{'}}{\lambda_m^{'}}}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)\overline{z^{m+1}}$$

for $\mu \ge 0,\ \ 0 \le t \le 1,\ \ 0 < \gamma \le m,\ \ |b_m| < \dfrac{1-\frac{\gamma}{m}}{1+2\mu(t-1)}$.

Corollary 7 Let $\mu \ge 0,\ \ 0 \le t \le 1,\ \ 0 < \gamma \le m, m \in \mathbb{N}$. If $f = h + \overline{g} \in \widetilde{H}(m)$ with h and g are of the form (1.3) belongs to the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then

$$\left\{\omega : |\omega| < 1 - \frac{m}{m+1} + \left(\frac{m(1+2\mu(t-1))}{(m+1)\left(1-\frac{\gamma}{m}\right)} - 1\right)|b_m|\right\} \subset f(\Delta).$$

Theorem 5 Let $\mu \ge 0,\ \ 0 \le t \le 1,\ \ 0 < \gamma \le m, m \in \mathbb{N}$ and let $\lambda_{m+1} \le \min\left(\frac{\lambda_n}{\lambda_m}, \frac{\lambda_n^{'}}{\lambda_m^{'}}\right),\ \ n \ge m+1$. If $f = h + \overline{g} \in \widetilde{H}(m)$, where h and g are of the form (1.3), belongs to the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then for $|z| = r < 1$,

$$|f(z)| \le (1+|b_m|)r^m + \frac{m}{(m+1)\lambda_{m+1}}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)r^{m+1}, \tag{4.3}$$

And $$|f(z)| \ge (1-|b_m|)r^m - \frac{m}{(m+1)\lambda_{m+1}}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)r^{m+1}. \tag{4.4}$$

The result is sharp.

Proof. Let $f \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then on using (2.1), from (1.3), we get for $|z| = r < 1$,

$$|f(z)| \le (1+|b_m|)r^m + \sum_{n=m+1}^{\infty}(|a_n|+|b_n|)r^n \le (1+|b_m|)r^m + r^{m+1}\sum_{n=m+1}^{\infty}(|a_n|+|b_n|)$$

$$\le (1+|b_m|)r^m + \frac{r^{m+1}}{\lambda_{m+1}}\sum_{n=m+1}^{\infty}\left(\frac{\lambda_n}{\lambda_m}|a_n| + \frac{\lambda_n'}{\lambda_m'}|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{(m+1)\lambda_{m+1}}\left(\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n| + \right.$$

$$\left.\sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n|\right)$$

$$\le (1+|b_m|)r^m + \frac{mr^{m+1}}{(m+1)\lambda_{m+1}}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)r^{m+1},$$

which proves (4.3). The result (4.4) can similarly be obtained. The bounds (4.3) and (4.4) are sharp for the function given by

$$f(z) = z^m + |b_m|\overline{z^m} + \frac{mr^{m+1}}{(m+1)\lambda_{m+1}}\left(1-\frac{1+2\mu(t-1)}{1-\frac{\gamma}{m}}|b_m|\right)\overline{z^{m+1}}$$

for $|b_m| < \dfrac{1-\frac{\gamma}{m}}{1+2\mu(t-1)}$.

Corollary 8 Let $\mu \ge 0$, $0 \le t \le 1$, $0 < \gamma \le m, m \in \mathbb{N}$ and let $\lambda_{m+1} \le \min\left(\frac{\lambda_n}{\lambda_m},\frac{\lambda_n'}{\lambda_m'}\right)$, $n \ge m+1$. If $f = h + \overline{g} \in \widetilde{H}(m)$, where h and g are of the form (1.3), belongs to the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then for $|z| = r < 1$, , then

$$\left\{\omega : |\omega| < 1 - \frac{m}{(m+1)\lambda_{m+1}} + \left(\frac{m(1+2\mu(t-1))}{(m+1)\left(1-\frac{\gamma}{m}\right)\lambda_{m+1}} - 1\right)|b_m|\right\} \subset f(\Delta).$$

5. EXTREME POINTS

In this section, we determine the extreme points for the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$.

Theorem 6 *let* $f = h + \overline{g} \in \widetilde{H}(m)$ *and*

$$h_m(z)=z^m,\ h_n(z)=z^m-\frac{m(m-\gamma)}{\left|m^2+\mu(n-m)(tn+m)\right|\frac{\lambda_n}{\lambda_m}}z^n\ (n\ge m+1),$$

$$g_n(z)=z^m+\frac{m(m-\gamma)}{\left|m^2+\mu(n+m)(tn-m)\right|\frac{\lambda_n'}{\lambda_m'}}\overline{z^n}\ (n\ge m),$$

then the function $f\in\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ if and only if it can be expressed as $f(z)=\sum_{n=m}^{\infty}(x_nh_n(z)+y_ng_n(z))$ where $x_n\ge 0, y_n\ge 0$ and $\sum_{n=m}^{\infty}(x_n+y_n)=1$. In particular, the extreme points of $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose that $f(z)=\sum_{n=m}^{\infty}(x_nh_n(z)+y_ng_n(z))$

Then,

$$f(z)=\sum_{n=m}^{\infty}(x_n+y_n)z^m-\sum_{n=m+1}^{\infty}\frac{m(m-\gamma)}{\left|m^2+\mu(n-m)(tn+m)\right|\frac{\lambda_n}{\lambda_m}}x_nz^n$$

$$+\sum_{n=m}^{\infty}\frac{m(m-\gamma)}{\left|m^2+\mu(n+m)(tn-m)\right|\frac{\lambda_n'}{\lambda_m'}}y_n\overline{z^n}$$

$$=z^m-\sum_{n=m+1}^{\infty}\frac{m(m-\gamma)}{\left|m^2+\mu(n-m)(tn+m)\right|\frac{\lambda_n}{\lambda_m}}x_nz^n+\sum_{n=m}^{\infty}\frac{m(m-\gamma)}{\left|m^2+\mu(n+m)(tn-m)\right|\frac{\lambda_n'}{\lambda_m'}}y_n\overline{z^n}$$

$\in\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$
by Theorem 2, since,

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}\left(\frac{m(m-\gamma)}{\left|m^2+\mu(n-m)(tn+m)\right|\frac{\lambda_n}{\lambda_m}}x_n\right)$$

$$+\sum_{n=m}^{\infty}\frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}\left(\frac{m(m-\gamma)}{\left|m^2+\mu(n+m)(tn-m)\right|\frac{\lambda_n'}{\lambda_m'}}y_n\right)$$

$$=\sum_{n=m+1}^{\infty}x_n+\sum_{n=m}^{\infty}y_n=1-x_m\le 1.$$

Conversely, let $f\in\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ and let

$$|a_n|=\frac{m(m-\gamma)x_n}{\left|m^2+\mu(n-m)(tn+m)\right|\frac{\lambda_n}{\lambda_m}}\text{ and }|b_n|=\frac{m(m-\gamma)y_n}{\left|m^2+\mu(n+m)(tn-m)\right|\frac{\lambda_n'}{\lambda_m'}}$$

and

$$x_m = 1 - \sum_{n=m+1}^{\infty} x_n - \sum_{n=m}^{\infty} y_n,$$

then, we get

$$f(z) = z^m - \sum_{n=m+1}^{\infty} |a_n| z^n + \sum_{n=m}^{\infty} |b_n| \overline{z}^n$$

$$= h_m(z) - \sum_{n=m+1}^{\infty} \frac{m(m-\gamma)x_n}{\left|m^2 + \mu(n-m)(tn+m)\right| \frac{\lambda_n}{\lambda_m}} x_n z^n$$

$$+ \sum_{n=m}^{\infty} \frac{m(m-\beta)y_n}{\left|m^2 + \lambda(n+m)(kn-m)\right| \frac{\lambda_n^{'}}{\lambda_m^{'}}} y_n \overline{z^n}$$

$$= h_m(z) + \sum_{n=m+1}^{\infty} (h_n(z) - h_m(z)) x_n + \sum_{n=m}^{\infty} (g_n(z) - h_m(z)) y_n$$

$$= h_m(z)\left(1 - \sum_{n=m+1}^{\infty} x_n - \sum_{n=m}^{\infty} y_n\right) + \sum_{n=m+1}^{\infty} h_n(z) x_n + \sum_{n=m}^{\infty} g_n(z) y_n$$

$$= \sum_{n=m}^{\infty} (x_n h_n(z) + y_n g_n(z)).$$

This proves the Theorem 6.

6 .Convolution and Convex Combinations

In this section, we show that the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ is invariant under convolution and convex combinations of its members.

Let the function $f = h + \overline{g} \in \widetilde{H}(m)$ where h and g are of the form (1.3) and

$$F(z) = z^m - \sum_{n=m+1}^{\infty} \frac{\lambda_n}{\lambda_m} |A_n| z^n + \sum_{n=m}^{\infty} \frac{\lambda_n^{'}}{\lambda_m^{'}} |B_n| \overline{z^n} \in \widetilde{H}(m). \tag{6.1}$$

The convolution between the functions of the class $\widetilde{H}(m)$ is defined by

$$(f * F)(z) = f(z) * F(z) = z^m - \sum_{n=m+1}^{\infty} \frac{\lambda_n}{\lambda_m} |a_n A_n| z^n + \sum_{n=m}^{\infty} \frac{\lambda_n^{'}}{\lambda_m^{'}} |b_n B_n| \overline{z^n}$$

Theorem 7 Let $\mu \geq 0$, $0 \leq t \leq 1$, $0 < \gamma \leq m, m \in \mathbb{N}$, if $f \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ and $F \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, then $f * F \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$.

Proof. Let $f = h + \overline{g} \in \widetilde{H}(m)$, where h and g are of the form (1.3) and $F \in \widetilde{H}(m)$ of the form (6.1) be in $\widetilde{R}_m^p([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k)$ class. Then by theorem (2), we have

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2 + \mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} |A_n| + \sum_{n=m}^{\infty} \frac{\left|m^2 + \mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} |B_n| \leq 1, \leq 1$$

which in view of (2.2), yields

$$|A_n| \le \frac{m(m-\gamma)}{\left|m^2+\mu(n-m)(tn+m)\right| \frac{\lambda_n}{\lambda_m}} \le \frac{m}{n} \le 1, n \ge m+1$$

$$|B_n| \le \frac{m(m-\beta)}{\left|m^2+\mu(n+m)(tn-m)\right| \frac{\lambda_n^{'}}{\lambda_m^{'}}} \le \frac{m}{n} \le 1, n \ge m.$$

Hence, by Theorem 2,

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} |a_n A_n| + \sum_{n=m}^{\infty} \frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} |b_n B_n|$$

$$\le \sum_{n=m+1}^{\infty} \frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} |a_n| + \sum_{n=m}^{\infty} \frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} |b_n|$$

≤ 1

which proves that $f * F \in \tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$.

We prove next that the class $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ is closed under convex combination of its members.

Theorem 8: Let $\mu \ge 0$, $0 \le t \le 1$, $0 < \gamma \le m, m \in \mathbb{N}$, the class $\tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$ is closed under convex combination.

Proof. Let $f_j \in \tilde{R}_m^p([(\nu_i),(\delta_i),\beta_i];\gamma;\mu,t)$, $j \in \mathbb{N}$ be of the form

$$f_j(z) = z^m - \sum_{n=m+1}^{\infty} |A_{j,n}| z^n + \sum_{n=m}^{\infty} |B_{j,n}| \overline{z^n}, j \in \mathbb{N}.$$

Then by Theorem 2, we have for $j \in \mathbb{N}$,

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} |A_{j,n}| + \sum_{n=m}^{\infty}\sum_{n=m}^{\infty} \frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} |B_{j,n}| \le 1. \qquad (6.2)$$

For some $0 \le t_j \le 1$, let $\sum_{j=1}^{\infty} t_j = 1$, the convex combination of $f_j(z)$ may be written as

$$\sum_{j=1}^{\infty} t_j \, f_j(z) = z^m - \sum_{n=m+1}^{\infty} \sum_{j=1}^{\infty} t_j |A_{j,n}| z^n + \sum_{n=m}^{\infty} \sum_{j=1}^{\infty} t_j |B_{j,n}| \overline{z^n}$$

Now by (6.2),

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} \sum_{j=1}^{\infty} t_j |A_{j,n}| + \sum_{n=m}^{\infty} \frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} \sum_{j=1}^{\infty} t_j |B_{j,n}|$$

$$= \sum_{j=1}^{\infty} t_j \left[\sum_{n=m+1}^{\infty} \frac{\left|m^2+\mu(n-m)(tn+m)\right|}{m(m-\gamma)} \frac{\lambda_n}{\lambda_m} |A_{j,n}| + \right.$$

$$\left. \sum_{n=m}^{\infty} \frac{\left|m^2+\mu(n+m)(tn-m)\right|}{m(m-\gamma)} \frac{\lambda_n^{'}}{\lambda_m^{'}} |B_{j,n}| \right] \le \sum_{j=1}^{\infty} t_j = 1$$

and so again by Theorem 2, we get $\sum_{j=1}^{\infty} t_j \ f_j(z) \in \widetilde{R}_m^p([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \gamma; \mu, t)$. This proves the result.

7 .Integral Operator

Now we examine a closure property of the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i]; \gamma; \mu, t)$ involving the generalized Bernardi Libera-Livingston Integral operator $L_{m,c}$ which is defined for $f = h + \overline{g} \in \widetilde{H}(m)$ by

$$L_{m,c}(f) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, \ c > -m, z \in \Delta. \tag{7.1}$$

Theorem 9 Let $\mu \geq 0$, $0 \leq t \leq 1$, $0 < \gamma \leq m, m \in \mathbb{N}$, if $f \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i]; \gamma; \mu, t)$, then $L_{m,c}(f) \in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i]; \gamma; \mu, t)$.

Proof. Let $f = h + \overline{g} \in \widetilde{H}(m)$, where h and g are of the form (1.3), belongs to the class $\widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i]; \gamma; \mu, t)$. Then, it follows from (7.1) that

$$\begin{aligned} L_{m,c}(f) &= z^m - \sum_{n=m+1}^{\infty}\left(\frac{c+m}{c+n}\right)|a_n|z^n + \sum_{n=m}^{\infty}\left(\frac{c+m}{c+n}\right)|b_n|\overline{z^n} \\ &\in \widetilde{R}_m^p([(\nu_i),(\delta_i),\beta_i]; \gamma; \mu, t) \end{aligned}$$

by (2.1), since,

$$\begin{aligned} &\sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}\left(\frac{c+m}{c+n}\right)|a_n| + \sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}\left(\frac{c+m}{c+n}\right)|b_n| \\ &\leq \sum_{n=m+1}^{\infty}\frac{|m^2+\mu(n-m)(tn+m)|}{m(m-\gamma)}\frac{\lambda_n}{\lambda_m}|a_n| + \sum_{n=m}^{\infty}\frac{|m^2+\mu(n+m)(tn-m)|}{m(m-\gamma)}\frac{\lambda_n'}{\lambda_m'}|b_n| \\ &\leq 1. \end{aligned}$$

This proves the result.

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